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# Existence of equilibrium in OLG economies with durable goods

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**Abstract**

We consider a standard pure exchange overlapping generations economy. The demographic structure consists of a new cohort of agents at each period with an economic activity extended over two successive periods. Our model incorporates durable goods that may be stored from one period to a successive period through a linear technology. In this model, we intend to study the mechanism of transfer between generations, and we show that the existence of an equilibrium can be established by considering an equivalent economy "*without*" durable goods, where the agents economic activity is extended over three successive periods.

*JEL classification*: C62, D50, D62.

**Keywords:** Overlapping generations model, durable goods, irreducibility, equilibrium, existence

## 1 Introduction

In a previous work [4], we considered a model of overlapping generations with production where increasing returns are allowed. The equilibrium existence rests on the following facts. There is a finite number of firms that are active forever but owned successively by 2-period lived individuals who receive exogeneous shares on firms at their birth. The producers are instructed to make non-negative profits while they choose a combination of prices and production plans. The consumers wealth consists of the value of their initial endowments and the profits they obtain from the firms according to their respective shares. They use all their wealth to pay for their lifetime consumption, leaving nothing

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to the next generations. Thus the transfers of property rights between generations are excluded.

We propose then to improve this work by studying the possibility of transfer of firms property across generations. This paper constitutes a first step to this improvement. We are working in the line of [8] who considers a model of overlapping generations with production. There are durable goods which depreciate over time, and they perpetuate the firms by allowing the transmission of the ownership shares accross generations through a stock market. However, we will first focus our attention to a pure exchange overlapping generations model as in [1, 2, 3], but in addition, commodities can be durable. We intend to study, in such a model, the mechanism of wealth transfer between generations, and we consider that the existence of durable goods makes the transfer of shares between generations meaningful.

We are in a model with infinitely many dates, and for simplicity, the set of commodities, perishable or durable, is the same at each date. Furthermore, there is no uncertainty, and every individual is supposed to correctly anticipate the future prices. There is a technology that permits to store and transfer the durable goods from one period to the next one. This storing technology is linear. The transfered good will implicitely act like an additionnal endowment at the date it is available. This technology is considered as a production function in the sense that a consumer who purchases a durable good can consume it and also use it as input to produce a good at the succeeding period. The relation between the spot prices in Proposition 1, which is like an arbitrage-free condition illustrates this, where price at each date can consist of two components: the input investment and the consumption cost. Malinvaud already considers this kind of phenomen in [9]. There, he introduces a forward market at each date, where agents can trade goods available only in the future.

The existence of durable goods in the model implies that agents in the end of their lifetime, will still own some goods which they will not need anymore in the next period. Thus we introduce, at each date a futures market that allows trades of goods available in a future date. This market helps the old generations to sell, at the end of their lifetime, their remaining durable goods after consumption. As mentionned above, this futures market is similar to a market of shares in a firm: at each date, old agents sell to the young the right to dispose of the remaining durable goods. This creates an additional resource to the old agents and a possibility for the young agents to increase their endowment when they become old. Purchase by the young on this market can be seen as savings that will finance the retirement of the old. The old generations, in return, will leave, at the end of their lifetime some commodities, to which the young generations, one period after, can have access. We can see those contracts as lifetime sale contracts called also "viager", where the old people can sell, for instance their house for an annuity, while they still occupy it until the end of their life, the buyers will then own the house right after. In general, durable goods can serve as collaterals in mortgage loans, in our case the reimbursement takes place one period after the agreement, during which the borrower seizes the collateral itself. We make an important assumption on the desirability of durable goods, which ensures that prices are positive, and in addition, there are no wastes at equilibrium: all the durable goods owned by the old will be bought by the young generations, that is all lendings will be payed back.

The model is descibed in Section 2. We make classical assumptions on the consumers, at the same level as for a pure exchange economy with perishable goods, but in addition we assume that goods are desirable. A first result of the paper establishes a relation between the spot prices and the futures prices at equilibrium.

Our main result is the existence of an equilibrium in this economy. The arbitrage-free conditions on equilibrium prices and the condition of no wastes allow us to consider a so-called "reduced equilibrium". This arbitrage-free condition at equilibrium also implies an indeterminacy concerning the purchase of young agents on the futures market. Indeed, thanks to the relations between spot and futures prices like no-arbitrage conditions, they will be indifferent between buying today on the futures market or buying tomorrow on the spot market.

We then establish the existence of the reduced equilibrium, for that we reformulate the model into an equivalent economy "without" durable goods as defined in Section 3. In this associated economy, all individuals will artificially live over three periods, and the consumption sets are transformed so that they will not consist of the positive orthant anymore. Furthermore the strong survival assumption is not satisfied since the initial endowments are no longer interior points.

Thus we establish the existence result in Section 4, where the proof is similar to Balasko et al in [1, 2, 3] but we also use the notion of irreducibility, as seen in [5, 6]. Irreducibility ensures that no matter how we allocate the individuals into two groups, each of the groups has some good for which the other group is willing to exchange with some goods of its own. This condition is easily obtained in our model thanks to the presence of durable and desirable goods and the connections between all the generations: they are indeed involved in a trade, either directly when they have common periods of life, or indirectly, in which case, individuals of each generation will successively act as intermediaries between them. The existence of equilibrium in the original model with durable goods follows the existence of equilibrium in the equivalent economy without durable goods.

# 2 The Model

We consider an overlapping generations economy with discrete and finitely many dates $(t = 1, 2 \ldots)$.

**Commodities**

There exists a finite set $\mathcal{L}$ of commodities available for consumption and trade in the world. We denote $\#\mathcal{L} = L$. Goods can be perishable or durable, and may suffer transformations from one period to an immediate successive period.

We represent these transformations by linear mappings $\Gamma^t : \mathbb{R}^L \to \mathbb{R}^L$ which transform each consumption $x_t \in \mathbb{R}^L_+$ at date $t$ into a bundle of goods $\Gamma^t(x_t) \in \mathbb{R}^L_+$ at date $t+1$. The commodity $\ell \in \mathcal{L}_t$ is perishable if $\Gamma^t(\delta_\ell) = 0$, where $\delta_\ell \in \mathbb{R}^L$ consists of one unit of commodity $\ell$ and nothing else.

So each good can be seen as a consumption good and an input if we think of $\Gamma^t$ as a production function.

**Consumers**

A generation 0, $\mathcal{I}_0$, lives only one period. At each period $t \in \mathbb{N}^*$, there is a finite and non-empty set of consumers $\mathcal{I}_t$ called generation $t$, who are born and live for two periods. We denote $\#\mathcal{I}_t = I_t$ and $\mathcal{I} = \cup_{t \in \mathbb{N}} \mathcal{I}_t$.

The consumption set of each individual $i \in \mathcal{I}_t$ is a subset $X^i = \mathbb{R}^L_+ \times \mathbb{R}^L_+$. The consumption set of consumers of generation 0 is $\mathbb{R}^L_+$.

Consumers preferences are represented by a utility function $u^i : X^i \to \mathbb{R}$.

The vector $e^i \in \mathbb{R}^L \times \mathbb{R}^L$ represents the initial endowment of the agent $i$ of the generation $t$.

**Assumption C.**

a) For all individuals in $\mathcal{I}$, $u^i$ is continuous, quasi-concave and locally non-satiated.

b) For all $t \in \mathbb{N}^*$, there exists $i_0(t) \in \mathcal{I}_t$ such that $u^{i_0(t)}$ is strictly monotonic.

Assumption C is a classical assumption in a standard finite economy.

**Assumption E.** For all $t \in \mathbb{N}^*$, for all $i \in \mathcal{I}_t$, $e^i \in \mathbb{R}^L_{++} \times \mathbb{R}^L_{++}$, for all $i \in \mathcal{I}_0$, $e^i \in \mathbb{R}^L_{++}$.

**Markets and Prices**

At each date $t$, there is a spot market for consumption. The price vector $p$ is an element of $\prod_{t=1}^{\infty} \mathbb{R}^L_+$ and $p_{t\ell}$ is the spot price of commodity $\ell$ at date $t$. Furthermore, we allow for trade between generations. To make clear how this trade takes place within one period, consider an individual born at date $t$ who purchases $x^i_{t+1}$ when old. This consumption gives right to $x^i_{t+1}$ at date $t+1$ and to $\Gamma^{t+1}(x^i_{t+1})$ available only at date $t+2$ that is, after his lifetime. So at the end of date $t+1$ he may wish to sell $\Gamma^{t+1}(x^i_{t+1})$ to young. We write $f^i_{t+1}$ the purchase of a young $i$ of generation $t+1$ from old agents of generation $t$ at date t+1, and $\Pi_{t+1}$ the vector price at which the trade is agreed, the future price vector $\Pi$ is an element of $\prod_{t=1}^{\infty} \mathbb{R}^L_+$. We remark that $f^i_{t+1}$ is not available before date $t+2$. This market can be seen as a futures market, where agents trade goods available only in the next period.

**Budget Constraints**

The budget constraint, for each agent $i \in \mathcal{I}_t$, $t \in \mathbb{N}^*$ is given by:

$$p_t \cdot x^i_t + p_{t+1} \cdot x^i_{t+1} + \Pi_t \cdot f^i_t \leq p_t \cdot e^i_t + p_{t+1} \cdot e^i_{t+1} + p_{t+1} \cdot \Gamma^t(x^i_t) + p_{t+1} \cdot \Gamma^t(f^i_t) + \Pi_{t+1} \cdot x^i_{t+1},$$

and for each agent $i \in \mathcal{I}_0$,

$$p_1 \cdot x^i_1 \leq p_1 \cdot e^i_1 + \Pi_1 \cdot x^i_1.$$

Each individual $i \in \mathcal{I}_t$ purchases $f^i_t$ at date $t$ from generation $t-1$, which is available only at date $t+1$ and gives right to $\Gamma^t(f^i_t)$. At date $t+1$, the same individual earns $\Gamma^t(x^i_t)$ from his previous consumption and sells $x^i_{t+1}$ on the futures market, when he is old, at the end of his lifetime.

We denote by $B^i(p, \Pi)$ the budget set of agent $i$ associated to $(p, \Pi)$.

**Feasibility conditions**

An allocation $((x^i), (f^i))$ in $\prod_{t=0}^{\infty} \prod_{i \in \mathcal{I}_t} X^i \times \prod_{t=0}^{\infty} \prod_{i \in \mathcal{I}_t} \mathbb{R}^L_+$ is feasible if:

$$\sum_{i \in \mathcal{I}_{t-1}} x^i_t = \sum_{i \in \mathcal{I}_t} f^i_t, \text{ for } t \geq 1,$$

$$\sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} x^i_t = \sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} e^i_t + \sum_{i \in \mathcal{I}_{t-1}} \Gamma^{t-1}(x^i_{t-1}) + \sum_{i \in \mathcal{I}_{t-1}} \Gamma^{t-1}(f^i_{t-1}), \text{ for } t > 1,$$

and

$$\sum_{i \in \mathcal{I}_0 \cup \mathcal{I}_1} x^i_1 = \sum_{i \in \mathcal{I}_0 \cup \mathcal{I}_1} e^i_1$$

The first equation indicates that there are no wastes, all the durable goods owned by old agents at the end of their lifetime will be bought by the young agents at that time.

We denote by $\mathcal{A}(\mathcal{E})$ the set of all feasible allocations.

**Equilibrium**

**Definition 1** An equilibrium of the economy $(\mathcal{E})$ is a list $(p^*, \Pi^*, (x^{i*}), (f^{i*}))$ in $\prod_{t=1}^{\infty} \mathbb{R}_+^L \times \prod_{t=1}^{\infty} \mathbb{R}_+^L \times \prod_{t=0}^{\infty} \prod_{i\in\mathcal{I}_t} X^i \times \prod_{t=0}^{\infty} \prod_{i\in\mathcal{I}_t} \mathbb{R}_+^L$ such that:
a) for all $t \geq 1$, for all $i \in \mathcal{I}_t$, $(x^{i*}, f^{i*})$ is a maximal element of $u^i$ satisfying the budget constraint:

$$p_t^* \cdot x_t^i + p_{t+1}^* \cdot x_{t+1}^i + \Pi_t^* \cdot f_t^i \leq p_t^* \cdot e_t^i + p_{t+1}^* \cdot e_{t+1}^i + p_{t+1}^* \cdot \Gamma^t(x_t^i) + p_{t+1}^* \cdot \Gamma^t(f_t^i) + \Pi_{t+1}^* \cdot x_{t+1}^i,$$

for all $i \in \mathcal{I}_0$, $x^{i*}$ is a maximal element of $u^i$ satisfying: $p_1^* \cdot x_1^i \leq p_1^* \cdot e_1^i + \Pi_1^* \cdot x_1^{i*}$.
b) the allocation $((x^{i*}), (f^{i*}))$ is feasible:

$$\sum_{i\in\mathcal{I}_{t-1}} x_t^{i*} = \sum_{i\in\mathcal{I}_t} f_t^{i*}, \text{ for } t \geq 1,$$

$$\sum_{i\in\mathcal{I}_{t-1}\cup\mathcal{I}_t} x_t^{i*} = \sum_{i\in\mathcal{I}_{t-1}\cup\mathcal{I}_t} e_t^i + \sum_{i\in\mathcal{I}_{t-1}} \Gamma^{t-1}(x_{t-1}^{i*}) + \sum_{i\in\mathcal{I}_{t-1}} \Gamma^{t-1}(f_{t-1}^{i*}), \text{ for } t > 1,$$

$$\sum_{i\in\mathcal{I}_0\cup\mathcal{I}_1} x_1^{i*} = \sum_{i\in\mathcal{I}_0\cup\mathcal{I}_1} e_1^i.$$

**Remark 1** The first equation in Condition *b) implies:*

$$\sum_{i\in\mathcal{I}_{t-1}\cup\mathcal{I}_t} x_t^{*i} = \sum_{i\in\mathcal{I}_{t-1}\cup\mathcal{I}_t} e_t^i + \sum_{i\in\mathcal{I}_{t-1}\cup\mathcal{I}_{t-2}} \Gamma^{t-1}(x_{t-1}^{i*}), \; \textit{for } t > 1$$

*This equation states that the consumptions at date $t$ involve consumptions of the preceeding generations.*

In the following, we denote by $\gamma^t$ the transpose of $\Gamma^t$.

**Proposition 1** *i) If $(p^*, \Pi^*)$ is an equilibrium price, then $(p^*, \hat{\Pi}^*)$ where $\hat{\Pi}_t^* := \gamma^t(p_{t+1}^*)$ for all $t$, is also an equilibrium.*

*ii) For all $t \in \mathbb{N}^*$, $p_t^* \gg \gamma^t(p_{t+1}^*)$.*

**Remark 2** *i)* The equilibrium price $(p^*, \hat{\Pi}^*)$ coincides with the equilibrium defined by Malinvaud in [9], where he considered an intertemporal economy with perishable commodities. These commodities do not cross time but may be available to agents only in future dates; in this case, agents are allowed to trade on forward markets at forward prices. Our model is then similar to the model à la Malinvaud with a production with constant returns.

*ii)* Furthermore, if we think $x_t^{i*}$ as a bundle that gives right to consumption at date $t$ as well as "input" for date $t+1$, we may write $p_t^* = \gamma^t(p_{t+1}^*) + p_t'$, where $\gamma^t(p_{t+1}^*)$ can be seen as the "input" cost while $p_t'$ is the consumption cost at date $t$.

*Proof.* At equilibrium, $\Pi_{t+1}^* \geq \gamma^{t+1}(p_{t+2}^*)$, for all $t$. Otherwise, all the young agents at date $t+1$ will have an arbitrage opportunity by buying a commodity $h$ on the futures market and reselling it on the spot market at date $t+2$. Then, there is no solution to the utility maximization problem under the budget constraint since the utility functions are locally nonsatiated.

Furthermore, if $\Pi_{t+1,h}^* > (\gamma^{t+1}(p_{t+2}))_h$, the young agents have no incentive to buy on the futures market because it is better to wait until the next period to make the purchase on the spot market, so $f_{t+1,h}^{i*} = 0$ for all $i$. But this implies that $x_{t+1,h}^{i*} = 0$ for all $i$. If we decrease the future price from $\Pi_{t+1,h}^*$ to $\hat{\Pi}_{t+1,h}^*$, then the budget set $B^i(p^*, \hat{\Pi}^*)$ associated to $(p^*, \hat{\Pi}^*)$ is smaller and included in the budget set $B^i(p^*, \Pi^*)$ associated to $(p^*, \Pi^*)$. Moreover, $((x^{i*}), (f^{i*}))$ belongs to $B^i(p^*, \hat{\Pi}^*)$. Hence as it is optimal in $B(p^*, \Pi^*)$, it is still optimal in the smaller set $B^i(p^*, \Pi^*)$. So $(p^*, \hat{\Pi}^*)$ is an equilibrium price with the same consumptions.

Now, let us suppose that there exists a durable commodity $h$ in $\mathcal{L}$ such that $p_{th}^* \leq (\gamma^t(p_{t+1}^*))_h$. Thus, either $p_{th}^* < (\gamma^t(p_{t+1}^*))_h$ or $p_{th}^* = (\gamma^t(p_{t+1}^*))_h$ If the first case holds, then young agents at date $t$, will have an arbitrage opportunity by buying the commodity $h$ at price $p_{th}^*$ on the spot market, and reselling it at the price $(\gamma^t(p_{t+1}^*))_h$ at date $t+1$. In the second case, the agent would be willing to buy as much as she wants of good $h$ when she is young, since her utility is locally nonsatiated, thus there would be no solution to the utility maximization problem. So necessarely, arbitrage-free condition implies $p_t^* \gg \gamma^t(p_{t+1}^*)$, for all $t \in \mathbb{N}^*$.
□

Thus the list $(p^*, (x^{i*}), (f^{i*}))$ in $\prod_{t=1}^{\infty} \mathbb{R}_+^L \times \prod_{t=0}^{\infty} \prod_{i \in \mathcal{I}_t} X^i \times \prod_{t=0}^{\infty} \prod_{i \in \mathcal{I}_t} \mathbb{R}_+^L$ such that:

*a)* for all $t \geq 1$, for all $i \in \mathcal{I}_t$, $x^{i*}$ is a maximal element of $u^i$ in the budget set:

$$\{x^i \in X^i \mid (p_t^* - \gamma^t(p_{t+1}^*)) \cdot x_t^i + (p_{t+1}^* - \gamma^{t+1}(p_{t+2}^*)) \cdot x_{t+1}^i \leq p_t^* \cdot e_t^i + p_{t+1}^* \cdot e_{t+1}^i\},$$

for all $i \in \mathcal{I}_0$, $x^{i*}$ is a maximal element of $u^i$ in the budget set:
$\{x^i \in X^i \mid (p_1^* - \gamma^1(p_2^*)) \cdot x_1^i \leq p_1^* \cdot e_1^i\}$.

*b)* the allocations $((x^{i*}), (f^{i*}))$ are feasible:

$$\sum_{i \in \mathcal{I}_{t-1}} x_t^{i*} = \sum_{i \in \mathcal{I}_t} f_t^{i*}, \text{ for } t \geq 1,$$

$$\sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} x_t^{*i} = \sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} e_t^i + \sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_{t-2}} \Gamma^{t-1}(x_{t-1}^{i*}), \text{ for } t > 1,$$

$$\sum_{i \in \mathcal{I}_0 \cup \mathcal{I}_1} x_1^{i*} = \sum_{i \in \mathcal{I}_0 \cup \mathcal{I}_1} e_1^i.$$

is an equilibrium with durable goods.

The budget constraint in *a)* indicates that the agents anticipate the future prices. Indeed the price vector $p_{t+2}^*$ appears in the budget constraint of the agents of generation $t$ because of the trade they make when old with the young of generation $t+1$ whose budget constraint involves prices at date $t+2$: at equilibrium these agents anticipate $\Pi_{t+1}$ to be equal to $\gamma^{t+1} p_{t+2}$.

Moreover, we note an indetermination for the $(f^{i*})$ given in *b)*. As a matter of fact, the individuals maximize their utility under a budget constraint that does not depend on the $f^{i*}$ anymore. The $f^{i*}$'s are only given by $\sum_{i \in \mathcal{I}_{t-1}} x_t^{i*} = \sum_{i \in \mathcal{I}_t} f_t^{i*}$, for $t \geq 1$, which means that the agents are indifferent between buying today on the futures market or buying tomorrow on the spot market.

To prove the existence of an equilibrium with durable goods, we will focus on the so-called "*reduced equilibrium*" defined as follows:

**Definition 2** A "*reduced equilibrium*" is an element $(p^*, (x^{i*}))$ of $\prod_{t=1}^{\infty} \mathbb{R}_+^L \times \prod_{t=0}^{\infty} \prod_{i \in \mathcal{I}_t} X^i$ such that:
*a)* for all $t \geq 1$, for all $i \in \mathcal{I}_t$, $x^{i*}$ is a maximal element of $u^i$ in the budget set:

$$\{x^i \in X^i \mid (p_t^* - \gamma^t(p_{t+1}^*)) \cdot x_t^i + (p_{t+1}^* - \gamma^{t+1}(p_{t+2}^*)) \cdot x_{t+1}^i \leq p_t^* \cdot e_t^i + p_{t+1}^* \cdot e_{t+1}^i\},$$

for all $i \in \mathcal{I}_0$, $x^{i*}$ is a maximal element of $u^i$ in the budget set:
$\{x^i \in X^i \mid (p_1^* - \gamma^1(p_2^*)) \cdot x_1^i \leq p_1^* \cdot e_1^i\}$.

*b)* the allocations $((x^{i*}), (f^{i*}))$ are feasible:

$$\sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} x_t^{*i} = \sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} e_t^i + \sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_{t-2}} \Gamma^{t-1}(x_{t-1}^{i*}), \text{ for } t > 1,$$

$$\sum_{i \in \mathcal{I}_0 \cup \mathcal{I}_1} x_1^{i*} = \sum_{i \in \mathcal{I}_0 \cup \mathcal{I}_1} e_1^i.$$

Indeed, if $(p^*, (x^{i*}))$ is a reduced equilibrium, then the list $(p^*, \Pi^*, (x^{i*}), (f^{i*}))$ where $\Pi_t^* := \gamma^t(p_{t+1}^*)$ for all $t$, and the $f^{i*}$'s are such that $\sum_{i\in\mathcal{I}_{t-1}} x_t^{i*} = \sum_{i\in\mathcal{I}_t} f_t^{i*}$, for $t \geq 1$, is an equilibrium.

Our main result is the following existence theorem.

**Theorem 1** *Under Assumptions C and E, the economy $\mathcal{E}$ has an equilibrium.*

# 3 An equivalent economy without durable goods

In the following, since the budget constraint of each individual involves prices over three periods of time, we build an *equivalent* economy $\tilde{\mathcal{E}}$ "without" durable goods, where each individual's lifetime is extended over three periods. This equivalent economy is similar to the standard pure exchange OLG model with perishable goods, and we will establish the existence of an equilibrium in $\tilde{\mathcal{E}}$ to prove the existence of an equilibrium in an economy with durable goods.

## 3.1 Description of the equivalent economy $\tilde{\mathcal{E}}$

We consider an overlapping generations model with discrete and infinitely many dates $t = 1, 2 \ldots$, and the same commodity space $\mathcal{L}$ at each date.
At each date $t$, the set of consumers, called generation $t$ is the same, and denoted by $\mathcal{I}_t$.

To describe the characteristics of the consumers we define the following linear mappings:
$\phi^t : (\mathbb{R}^L)^2 \to (\mathbb{R}^L)^3$, for $t \geq 1$, by $\phi^t(x_t^i, x_{t+1}^i) = (x_t^i, \xi_{t+1}^i, \zeta_{t+2}^i)$, with
$\xi_{t+1}^i = x_{t+1}^i - \Gamma^t(x_t^i)$, and $\zeta_{t+2}^i = -\Gamma^{t+1}(\xi_{t+1}^i + \Gamma^t(x_t^i))$,
$\phi^0 : \mathbb{R}^L \to (\mathbb{R}^L)^2$, by $\phi^0(x_1^i) = (\xi_1^i, \zeta_2^i)$, with $\xi_1^i = x_1^i$, and $\zeta_2^i = -\Gamma^1(\xi_1^i)$.

The consumption sets are now defined as follows:
For each $i \in \mathcal{I}_0$, $\tilde{X}^i := \phi^0(\mathbb{R}_+^L)$, and for each $i \in \mathcal{I}_t$, $t \geq 1$, $\tilde{X}^i := \phi^t((\mathbb{R}_+^L)^2)$.

Thus the consumption set of an agent $i$ of generation $t$, $\tilde{X}^i$ is defined over three periods: $t$, $t+1$ and $t+2$, and that of generation 0 is now defined over two periods: $t = 1$ and $t = 2$.

The initial endowment is defined as follows:
$\tilde{e}^i = (e_1^i, 0) \in \mathbb{R}_{++}^L \times \mathbb{R}^L$, for $i \in \mathcal{I}_1$, and $\tilde{e}^i = (e_t^i, e_{t+1}^i, 0) \in \mathbb{R}_{++}^L \times \mathbb{R}_{++}^L \times \mathbb{R}^L$, for $i \in \mathcal{I}_t$.

For each $t$, the function $\phi^t$ is injective, the function $\phi^t_{|(\mathbb{R}^L_+)^2} : (\mathbb{R}^L_+)^2 \to \tilde{X}^i$ is bijective. Its inverse $\psi^t : \tilde{X}^i \to (\mathbb{R}^L_+)^2$ is thus well defined. In the same way, the function $\phi^0_{|\mathbb{R}^L_+} : \mathbb{R}^L_+ \to \tilde{X}^i$ is bijective, and its inverse $\psi^0 : \tilde{X}^i \to \mathbb{R}^L_+$ is thus also well defined.

We can now define the new utility functions $\tilde{u}^i : \tilde{X}^i \to \mathbb{R}$, by $\tilde{u}^i = u^i \circ \psi^t$. Note that for each $i \in \mathcal{I}_0$, $\tilde{u}^i(\xi^i_1, \zeta^i_2) = u^i(\xi^i_1)$, and for each $i \in \mathcal{I}_t$, $t \geq 1$, $\tilde{u}^i(x^i_t, \xi^i_{t+1}, \zeta^i_{t+2}) = u^i(x^i_t, \xi^i_{t+1} + \Gamma^t(x^i_t))$.

The definition below coincides with the standard definition of a competitive equilibrium in an OLG economy without durable goods.

**Definition 3** An equilibrium in $\tilde{\mathcal{E}}$ is a list $(p^*, (\chi^{i*}))$ in $\prod_{t=1}^\infty \mathbb{R}^L_+ \times \prod_{t=0}^\infty \prod_{i\in\mathcal{I}_t} \tilde{X}^i$, such that:
a) for all $t \in \mathbb{N}$, for all $i \in \mathcal{I}_t$, $\chi^{i*} = (x^{i*}_t, \xi^{i*}_{t+1}, \zeta^{i*}_{t+2})$ is a maximal element of $\tilde{u}^i$ satisfying the equivalent budget constraint:
$p^*_t \cdot x^{i*}_t + p^*_{t+1} \cdot \xi^{i*}_{t+1} + p^*_{t+2} \cdot \zeta^{i*}_{t+2} \leq p^*_t \cdot \tilde{e}^i_t + p^*_{t+1} \cdot \tilde{e}^i_{t+1}$
b) the consumption plan $(\chi^{i*})$ is feasible:
$\sum_{i\in\mathcal{I}_0} \xi^{i*}_1 + \sum_{i\in\mathcal{I}_1} x^{i^*}_1 = \sum_{i\in\mathcal{I}_0\cup\mathcal{I}_1} \tilde{e}^i_1$
$\sum_{i\in\mathcal{I}_{t-2}} \zeta^{i*}_t + \sum_{i\in\mathcal{I}_{t-1}} \xi^{i*}_t + \sum_{i\in\mathcal{I}_t} x^{i*}_t = \sum_{i\in\mathcal{I}_{t-1}\cup\mathcal{I}_t} \tilde{e}^i_t,\ t > 1$

**Proposition 2** *If $(p^*, (\chi^{i*}))$, where $\chi^{i*} = (x^{i*}_t, \xi^{i*}_{t+1}, \zeta^{i*}_{t+2})$ for $i \in \mathcal{I}_t$ is an equilibrium of the equivalent economy, then $(p^*, (x^{i*}))$ is a reduced equilibrium, where $x^{i*} = (x^{i*}_t, \xi^{i*}_{t+1} + \Gamma(x^{i*}_t))$ for $i \in \mathcal{I}_t$, $t \geq 1$.*

*Proof.* Indeed, by construction, if $(\chi^{i*})$ is feasible in $\tilde{\mathcal{E}}$, then, $((x^{i*}))$ defined by $x^{i*} = (x^{i*}_t, \xi^{i*}_{t+1} + \Gamma(x^{i*}_t))$ is feasible, that is:

$$\sum_{i\in\mathcal{I}_0} \xi^{i*}_1 + \sum_{i\in\mathcal{I}_1} x^{i^*}_1 = \sum_{i\in\mathcal{I}_0\cup\mathcal{I}_1} x^{i^*}_1 = \sum_{i\in\mathcal{I}_0\cup\mathcal{I}_1} \tilde{e}^i_1$$

$$\sum_{i\in\mathcal{I}_{t-2}} \zeta^{i*}_t + \sum_{i\in\mathcal{I}_{t-1}} \xi^{i*}_t + \sum_{i\in\mathcal{I}_t} x^{i*}_t = \sum_{i\in\mathcal{I}_{t-2}} -\Gamma^{t-1}(x^{i*}_{t-1}) + \sum_{i\in\mathcal{I}_{t-1}} (x^{i*}_t - \Gamma^{t-1}(x^{i*}_{t-1})) + \sum_{i\in\mathcal{I}_t} x^{i*}_t$$

Thus:

$$\sum_{i\in\mathcal{I}_{t-1}\cup\mathcal{I}_t} x^{*i}_t = \sum_{i\in\mathcal{I}_{t-1}\cup\mathcal{I}_t} \tilde{e}^i_t + \sum_{i\in\mathcal{I}_{t-1}\cup\mathcal{I}_{t-2}} \Gamma^{t-1}(x^{i*}_{t-1}),\ \text{for } t > 1$$

Furthermore the optimality of $\chi^{i*}$ for the utility function $\tilde{u}^i$ under the equivalent budget constraint above implies, by construction, the optimality of $x^{i*} = (x^{i*}_t, \xi^{i*}_{t+1} + \Gamma(x^{i*}_t))$ for the utility function $u^i$ and under the budget constraint:

$$p_t^* \cdot x_t^i + p_{t+1}^* \cdot \underbrace{(x_{t+1}^i - \Gamma^t(x_t^i))}_{\xi_{t+1}^i} + p_{t+2}^* \cdot \underbrace{(-\Gamma^{t+1}(x_{t+1}^i))}_{\zeta_{t+2}^i} \leq p_t^* \cdot e_t^i + p_{t+1}^* \cdot e_{t+1}^i$$

$$p_1^* \cdot \underbrace{x_1^i}_{\xi_1^i} + p_2^* \cdot \underbrace{(-\Gamma^1(x_1^i))}_{\zeta_2^i} \leq p_1^* \cdot x_1^i,\ i \in \mathcal{I}_0.$$

□

### 3.2 Some properties of the equivalent economy

- For all $t = 1, 2, \ldots$, for all $i \in \mathcal{I}_t$, the consumption sets $\tilde{X}^i$ are non-empty, closed, and convex.
  We note that the $\tilde{X}^i$'s are not the positive orthants, but the consumptions at date $t$ for $i \in \mathcal{I}_t$ are bounded from below. Indeed, for each individual $i$ of generation $t$, we allow for negative consumptions at dates $t+1$ and $t+2$, but the consumptions at date $t+1$ and $t+2$ are constrained by the consumptions at the preceeding dates. These kind of consumption sets are considered by Florenzano in [7], in which nonnegative components are called consumptions and nonpositive ones deliveries: the activity of an agent of generation $t$ at date $t+1$ artificially consists of delivering the remaining goods she holds at this period to agents of the next genaration.

- For all $t = 1, 2, \ldots$, for all $i \in \mathcal{I}_t$, the utility functions $\tilde{u}^i$ defined above inherit the conditions on $u^i$ in Assumption C, thanks to the linearity of $\phi^t$ and $\psi^t$. In particular, there exists $i_0(t)$ in $\mathcal{I}_t$ such that $\tilde{u}^{i_0}$ is strictly monotonic with respect to the two first variables $(x_t^{i_0}, \xi_{t+1}^{i_0})$.

- The set of feasible allocations $\mathcal{A}(\tilde{\mathcal{E}})$, that is the set of allocations satisfying the market clearing condition (Condition (b) of Definition 3) for the economy $\tilde{\mathcal{E}}$ is a subset of a compact set for the product topology. Indeed, let $e \in \prod_{t\in\mathbb{N}^*} \mathbb{R}_+^L$ be defined by $e_t = \sum_{i\in\mathcal{I}_t\cup\mathcal{I}_{t-1}} e_t^i$. Let $e' \in \prod_{t\in\mathbb{N}^*} \mathbb{R}_+^L$ such that $e' \geq e$. Then, there exists a sequence of nonnegative vectors $(M_t)_{t\geq 1}$ such that for all $(\chi^i) \in \mathcal{A}(\tilde{\mathcal{E}})$, with $\chi^i = (x_t^i, \xi_{t+1}^i, \zeta_{t+2}^i)$, $i \in \mathcal{I}_t$, for all $t = 1, 2\ldots$, we have:
  $0 \leq x_t^i \leq M_t$, $-\Gamma^t(M_t) \leq \xi_{t+1}^i \leq M_{t+1}$, and $-\Gamma^{t+1}(M_{t+1}) \leq \zeta_{t+2}^i \leq 0$.
  $M_t$ is recursively defined by: $M_t = e'_t + \Gamma^{t-1}(M_{t-1} + \Gamma^{t-2}(M_{t-2}) \cdots + \Gamma^1(M_1))$, where $M_1 = e'_1$. (See Appendix)

# 4 Existence of equilibrium in the equivalent economy

To establish the existence of equilibrium in $\tilde{\mathcal{E}}$, we first truncate the economy $\tilde{\mathcal{E}}$ at a finite horizon $\tau$ and consider the set of all individuals born up to period $\tau - 2$, $\mathcal{I}_0^{\tau-2} = \cup_{t=0}^{\tau-2} \mathcal{I}_t$.

For each $i \in \mathcal{I}_0$,
$\tilde{X}^{\tau i} = \{a \in (\mathbb{R}_+^L)^\tau \mid (a_1, a_2) \in \tilde{X}^i, a_{t'} = 0, \forall t' > 2\}$
$\tilde{u}^{\tau i}(a) = \tilde{u}^i(a_1, a_2)$
$\tilde{e}^{\tau i} = (\tilde{e}_{t'}^{\tau i})_{t'=1}^\tau$ such that $\tilde{e}_1^{\tau i} = \tilde{e}_1^i$, and $\tilde{e}_{t'}^{\tau i} = 0$ if $t' > 1$.
For each $i \in \mathcal{I}_t, t = 1, 2 \ldots \tau - 3$,
$\tilde{X}^{\tau i} = \{a \in (\mathbb{R}_+^L)^\tau \mid (a_t, a_{t+1}, a_{t+2}) \in \tilde{X}^i,\ a_{t'} = 0, \forall t' \neq t, t+1, t+2\}$
For each $i \in \mathcal{I}_{\tau-2}$,
$\tilde{X}^{\tau i} = \{a \in (\mathbb{R}_+^L)^\tau \mid (a_{\tau-2}, a_{\tau-1}, -\Gamma^{\tau-1}(a_{\tau-1} + \Gamma^{\tau-2}(a_{\tau-2})) \in \tilde{X}^i,$
$a_{t'} = 0, \forall t' \neq \tau - 1, \tau - 2\}$
$\tilde{u}^{\tau i}(a) = \tilde{u}^i(a_t, a_{t+1}, a_{t+2})$
$\tilde{e}^{\tau i} = (\tilde{e}_{t'}^{\tau i})_{t'=1}^\tau$ such that $\tilde{e}_t^{\tau i} = \tilde{e}_t^i$, $\tilde{e}_{t+1}^{\tau i} = \tilde{e}_{t+1}^i$ and $\tilde{e}_{t'}^{\tau i} = 0$ if $t' \neq t, t+1$.

We note that the standard survival assumption is not satisfied because the initial endowment $\tilde{e}^{\tau i}$ may not belong to the consumption set $\tilde{X}^{\tau i}$. Indeed, for $i \in \mathcal{I}_t$, $(\tilde{e}_t^i, \tilde{e}_{t+1}^i, \tilde{e}_{t+2}^i)$ may not be in $\tilde{X}^i$ since $\tilde{e}_{t+2}^i = 0 \neq -\Gamma^{t+1}(\tilde{e}_{t+1}^i + \Gamma^t(\tilde{e}_t^i))$ if $\Gamma^{t+1} \neq 0$. So in order to overcome this difficulty, we work with a free-disposal equilibrium by introducing the free disposal cone $Y := -(\mathbb{R}_+^L)^\tau$. Then $\tilde{e}^{\tau i} \in \tilde{X}^{\tau i} - Y$. See [5] and [6] for the existence of free-disposal equilibrium in a pure exchange economy.

Now, we introduce a weak notion of equilibrium, called pseudo-equilibrium, in which we do not require the market clearing condition at periods $\tau - 1$ and $\tau$. Indeed, the truncation of an equilibrium is not an equilibrium but a pseudo-equilibrium. (See Lemma 1 below).

**Definition 4** A pseudo-equilibrium in the truncated economy $\tilde{\mathcal{E}}_\tau$ is an element $(p^*, (a^{i*}) \in (\mathbb{R}_+^L)^\tau \times \prod_{i \in \mathcal{I}_0^{\tau-2}} X^{\tau i}$ such that:

a) for all $t = 1, 2 \ldots \tau - 2$, for all $i \in \mathcal{I}_t$, $a^{i*}$ is a maximal element of $\tilde{u}^{\tau i}$ in the budget set

$$\{a^i \in \tilde{X}^{\tau i} \mid p^* \cdot a_i \leq p^* \cdot \tilde{e}^{\tau i}\};$$

for all $i \in \mathcal{I}_0$, $a^{i*}$ is a maximal element of $\tilde{u}^{\tau i}$ in the budget set $\{a^i \in \tilde{X}^{\tau i} \mid p^* \cdot a^i \leq p^* \cdot \tilde{e}^{\tau i}\}$;

b) For all $t = 1, \dots, \tau - 2$,

$$\sum_{i \in \mathcal{I}_0^{\tau-2}} a_t^{i*} = \sum_{i \in \mathcal{I}_0^{\tau-2}} \tilde{e}_t^{\tau i},$$

$$\sum_{i \in \mathcal{I}_0^{\tau-2}} a_{\tau-1}^{i*} \leq \sum_{i \in \mathcal{I}_0^{\tau-2}} \tilde{e}_{\tau-1}^{\tau i} + \sum_{i \in \mathcal{I}_{\tau-1}} \tilde{e}_{\tau-1}^{i}.$$

According to this definition, at period $\tau - 1$, we artificially increase the initial endowments by adding those of the consumers of the generation $\tau - 1$.

**Lemma 1** *If $\bar{\tau} > \tau$ and $(\bar{p}^*, (\bar{a}^{i*}))$ is a pseudo-equilibrium in the economy $\mathcal{E}_{\bar{\tau}}$, then the price and the allocations restricted to the $\tau - 1$ first periods $\left(\hat{p}^*, (\hat{a}^{i*})_{i \in \mathcal{I}_0^{\tau-2}}\right)$ defined by*

$\hat{p}^* = (\bar{p}_t^*)_{t=1}^{\tau-1}$,
$\hat{a}^{i*} = (\bar{a}_t^{i*})_{t=1}^{\tau-1}$, *for all $i \in \mathcal{I}_0^{\tau-2}$, is a pseudo-equilibrium in the economy $\mathcal{E}_\tau$.*

In the following, we will establish the existence of a pseudo-equilibrium in $\tilde{\mathcal{E}}_\tau$. For that, we use the fact that an equilibrium with free-disposal is a pseudo-equilibrium. But since $\tilde{e}^{\tau i} \in \tilde{X}^{\tau i} - Y$, the problem of non-interiority of the initial endowments leads us to first make use of the notion of quasi-equilibrium with free-disposal as an intermediate step.

**Definition 5** A quasi-equilibrium with free-disposal in $\tilde{\mathcal{E}}_\tau$ is a list $(p^*, (a^{i*}), y^*)$ in $(\mathbb{R}_+^L)^\tau \times \prod_{i \in \mathcal{I}_0^{\tau-2}} X^{\tau i} \times Y$ such that:
a') for all $t = 1, 2 \dots \tau - 1$, $a^{i*}$ is an element of the budget set:

$$\{a^i \in \tilde{X}^{\tau i} \mid p^* \cdot a_i \leq p^* \cdot \tilde{e}^{\tau i}\}$$

and for all $a^i \in \tilde{X}^{\tau i}$ such that: $p^* \cdot a_i < p^* \cdot \tilde{e}^{\tau i}$, $\tilde{u}^{\tau i}(a^i) \leq \tilde{u}^{\tau i}(a^{i*})$,

for all $i \in \mathcal{I}_0$, $a^{i*} \in \{a^i \in \tilde{X}^i \mid p^* \cdot a^i \leq p^* \cdot \tilde{e}^{\tau i}\}$ and for all $a^i \in \tilde{X}^{\tau i}$ such that $p^* \cdot a^i < p^* \cdot \tilde{e}^{\tau i}$, $\tilde{u}^{\tau i}(x^i) \leq \tilde{u}^{\tau i}(x^{i*})$,
b) $p^* \cdot y \leq p^* \cdot y^* = 0$ for all $y \in Y$
c) $\sum_{i \in \mathcal{I}_0^{\tau-2}} a^{i*} = \sum_{i \in \mathcal{I}_0^{\tau-2}} \tilde{e}^{\tau i} + y^*$
d) $p^* \neq 0$.

**Proposition 3** *For all $\tau \geq 3$, $\tilde{\mathcal{E}}_\tau$ has a quasi-equilibrium with free-disposal $(p^*, (a^{i*}), (y*)) \in (\mathbb{R}_+^L)^\tau \times \prod_{i \in \mathcal{I}_0^{\tau-2}} X^{\tau i} \times Y$.*

*Proof*

Indeed, $\tilde{\mathcal{E}}_\tau$ satisfies all the necessary conditions of existence of quasi-equilibrium in an exchange economy where free-disposal activities are possible. [See Florenzano in [6], Proposition 2.2.2]

- $\tilde{e}^{\tau i} \in \tilde{X}^{\tau i} - Y$ and $\sum_{i \in \mathcal{I}_0^{\tau-2}} \tilde{e}^{\tau i} \in int \left( \sum_{i \in \mathcal{I}_0^{\tau-2}} \tilde{X}^{\tau i} - Y \right)$
- for all $i$, $\tilde{u}^{\tau i}$ satisfies the classical conditions of continuity, quasi-concavity and local non-satiation,
- the set of feasible allocations $\mathcal{A}(\tilde{\mathcal{E}}_\tau)$ is a subset of a compact set of $(\mathbb{R}^L)^\tau$. $\square$

Actually, one way to go from a quasi-equilibrium to an equilibrium is the notion of McKenzie-Debreu irreducibility. But following Assumption C and Assumption D made on the original economy, we establish that the truncated economy $\tilde{\mathcal{E}}_\tau$ is McKenzie-Debreu irreducible.

**Proposition 4** *The truncated economy $\tilde{\mathcal{E}}_\tau$, equiped with the disposal activity $Y$ is McKenzie-Debreu irreducible, that is for all non-empty disjoint subsets $J_1, J_2$ of $\mathcal{I}_0^{\tau-2}$, $J_1, J_2 \neq \mathcal{I}_0^{\tau-2}$, $\mathcal{I}_0^{\tau-2} = J_1 \sqcup J_2$,and for all feasible allocation $(a^i) \in \prod_{i \in \mathcal{I}_0^{\tau-2}} X^{\tau i}$, there exists an allocation $(a'^i) \in \prod_{i \in \mathcal{I}_0^{\tau-2}} X^{\tau i}$ such that:*
*1- $\tilde{u}^{\tau i}(a'^i) \geq \tilde{u}^{\tau i}(a^i)$ for all $i \in J_1$ and $\exists j \in J_1, \tilde{u}^{\tau j}(a'^j) > \tilde{u}^{\tau j}(a^j)$,*
*2- $\sum_{i \in \mathcal{I}_0^{\tau-2}}(a'^i - \tilde{e}^{\tau i}) - \sum_{i \in J_2}(\tilde{e}^{\tau i} - a^i) \in Y$*

Taking into account the feasibility of the allocation $(a^i)$, condition *2* can also be written as: $\sum_{i \in \mathcal{I}_0^{\tau-2}} a'^i - \sum_{i \in J_1} a^i + \sum_{i \in J_2} \tilde{e}^{\tau i} \in Y$.

The irreducibility condition says that whenever the individuals are allocated into two nonempty and disjoint groups $J_1$ and $J_2$, then for any feasible allocation $(a^i)$ and after disposing of any eventual surplus, $\sum_{i \in \mathcal{I}_0^{\tau-2}} \tilde{e}^{\tau i} + \sum_{i \in J_2}(\tilde{e}^{\tau i} - x^i)$ can be allocated to group $J_1$ improving the situation of its members as given by Relation 1.

*Proof.*

First case, there exists $t$ such that $\mathcal{I}_t \cap J_1 \neq \emptyset$, and $\mathcal{I}_t \cap J_2 \neq \emptyset$. So let $i_1$ and $i_2$ be in $\mathcal{I}_t$ such that $i_1 \neq i_2$ and $i_1 \in J_1$, $i_2 \in J_2$. Since $\tilde{e}_t^{\tau i}$ and $\tilde{e}_{t+1}^{\tau i}$ are positive for $i = i_1, i_2$, each one is able to provide some good for which the other one is willing to exchange with some good of its own thanks to Assumption C. For instance, take $a'^{i_1} = a^{i_1} + \epsilon$ where $\epsilon > 0$ is arbitrarely small, $a'^{i_2} = \tilde{e}^{\tau i_2} - \epsilon \gg 0$ for $\epsilon$ small enough, $a'^i = a^i$, for $i \in J_1, i \neq i_1$, and $a'^i = \tilde{e}^{\tau i}$, for $i \in J_2, i \neq i_2$. Clearly, $a'^i$ satisfies Relations 1 and 2: the situation of one group will be moved to a preferred position, by adding a feasible trade from the other group.

Suppose now that there does not exist $t$ such that $\mathcal{I}_t \cap J_1 \neq \emptyset$ and $\mathcal{I}_t \cap J_2 \neq \emptyset$; let us define: $\bar{t}_1 := max\{t \mid \mathcal{I}_t \subset J_1\}$, $\bar{t}_2 := max\{t \mid \mathcal{I}_t \subset J_2\}$.
Note that the sets $\{t \mid \mathcal{I}_t \subset J_1\}$ and $\{t \mid \mathcal{I}_t \subset J_2\}$ are disjoint and thei union is $\{1, 2, \ldots \tau - 2\}$.

If $\bar{t}_1 \neq \tau - 2$, then $\mathcal{I}_{\bar{t}_1} \subset J_1$ and $\mathcal{I}_{\bar{t}_1+1} \subset J_2$. If $\bar{t}_1 = \tau - 2$, then since $\bar{t}_2 \neq \tau - 2$, $\mathcal{I}_{\bar{t}_2} \subset J_2$ and $\mathcal{I}_{\bar{t}_2+1} \subset J_1$. Since the two sub-cases can be treated similarly, we deal only with the first one, in which $\mathcal{I}_{\bar{t}_1} \subset J_1$ and $\mathcal{I}_{\bar{t}_1+1} \subset J_2$. In this sub-case, since $\tilde{e}^{\tau i}_{\bar{t}_1}, \tilde{e}^{\tau i}_{\bar{t}_1+1}$ are positive for $i \in \mathcal{I}_{\bar{t}_1}$, as well as $\tilde{e}^{\tau i}_{\bar{t}_1+1}$ for $i \in \mathcal{I}_{\bar{t}_1+1}$, both generations are able to provide some commodity during their common period of life. In particular, consider the individual $i_0(\bar{t}_1)$ mentionned in Assumption C, then a young individual $i$ of generation $\bar{t}_1 + 1$ can provide some goods to $i_0(\bar{t}_1)$, improving the utility of $i_0(\bar{t}_1)$ when he is old at date $\bar{t}_1 + 1$. If we keep the allocations of the other individuals of $J_1$ unchanged, and let the other members of $J_2$ consume their initial endowments, Relations 1 and 2 are satisfied. $\square$

Thanks to Assumptions C and D on the original economy, the McKenzie-Debreu irreducibility of the truncated economy $\tilde{\mathcal{E}}_\tau$ and the interiority of the total initial endowment, we get that a quasi-equilibrium with free-disposal $(p^*, (a^{i*})) \in \prod_{t=1}^{\tau} \mathbb{R}^L_+ \times \prod_{i \in \mathcal{I}_0^{\tau-2}} X^{\tau i}$ is an equlibrium with free-disposal. (See Florenzano [6], Proposition 2.3.2 and Corollary 2.3.2)

**Remark 3** The strict monotonicity of the utility function $u^{i_0}$ of an individual $i_0$ in $\mathcal{I}_t$ in Assumption C implies that $p^*_t \gg 0$ for all $t = 1, 2, \ldots \tau - 1$, thus $y^*_t = 0$ for all $t = 1, 2, \ldots \tau - 1$.

Thus, since the equilibrium is realized without disposal of surplus, we get that the equilibrium with free-disposal is actually an equilibrium so a pseudo-equilibrium. Hence, one finally obtains:

**Proposition 5** *For all $\tau \geq 3$, there exists a pseudo-equilibrium of the economy $\tilde{\mathcal{E}}_\tau$.*

The following lemma gives properties of the pseudo-equilibrium. We normalize a non zero equilibrium price $p^*$ so that $\sum_{t=1}^{\tau} \sum_{\ell \in \mathcal{L}} p^*_{t\ell} = 1$.

**Lemma 2** *If $(p^*, (a^{i*})) \in (\mathbb{R}^L_+)^\tau \times \prod_{i \in \mathcal{I}_0^{\tau-}} X^{\tau i}$ is a pseudo-equilibrium, then $p^*_t \gg 0$, for all $t$.*
*Furthermore, the set of pseudo-equilibria of the economy $\tilde{\mathcal{E}}_\tau$ with a normalized price is closed.*

*Proof:* See Appendix.

The last step of the existence of equilibrium in the reduced economy consists of considering a sequence of pseudo-equilibria in the truncated economy with an horizon increasing to infinity. We follow [4], and establish that the sequence of equilibrium prices in the truncated economies remains in a compact set for the product topology on $\prod_{t=1}^{\infty} \mathbb{R}^L$.

From the previous section, for all $T \geq 2$, there exists a pseudo-equilibrium $(p^T, (a^{iT}))$ of the economy $\tilde{\mathcal{E}}_T$. Since we have previously proved that $p_1^T \neq 0$, we normalize $p^T$ so that $\sum_{\ell \in \mathcal{L}} p_{1\ell}^T = 1$.

We extend the price and the allocations to the whole space $\prod_{t=1}^{\infty} \mathbb{R}^L$ by adding zeros for the missing components without modifying the notations. So, now the sequences $(p^T)$, $(a^{iT})$ are in $\prod_{t=1}^{\infty} \mathbb{R}^L$.

**Lemma 3** *For all $t$, there exists $\tilde{k}_t \in \mathbb{R}_+$ such that for all $T$, $0 \leq p_t^T \leq \tilde{k}_t \mathbf{1}$, where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^L$.*

*Proof.* See Appendix.

Now we show that the sequence of pseudo-equilibria remains in a compact set and we prove that a cluster point is an equilibrium of the OLG economy $\mathcal{E}$.

From the compactness of $\mathcal{A}(\tilde{\mathcal{E}})$ and the above lemma, the sequence of $T$-equilibrium of the economy $\mathcal{E}_T$, $(p^T, (a^{iT}))$, remains in a compact set for the product topology of $\prod_{t=1}^{\infty} \mathbb{R}^L \times \prod_{t'=1}^{\infty} \prod_{i \in \mathcal{I}_{t'}} \prod_{t=1}^{\infty} \mathbb{R}^L$. Since this is a countable product of finite dimensional spaces, the product topology is metrizable on the compact sets and there exists a sub-sequence $(p^{T^\nu}, (a^{iT^\nu}))$ of $(p^T, (a^{iT}))$, which converges to $(p^*, (a^{i*}))$. We recall that the convergence for the product topology implies the usual convergence when we consider only a finite number of components.

For each $\tau \geq 3$, for $\nu$ large enough, the restriction of $(p^{T^\nu}, (a^{iT^\nu}))$ to the $\tau$ first periods is a pseudo-equilibrium of $\tilde{\mathcal{E}}_\tau$ (see Lemma 1) and it converges to the restriction of $(p^*, (a^{i*}))$ to the $\tau$ first periods. From Lemma 2, this restriction is a pseudo-equilibrium of $\tilde{\mathcal{E}}_\tau$. From Definition 4 and the notations above, one deduces that $(p^*, (\alpha^{i*}))$ defined as follows is an equilibrium for the OLG economy $\tilde{\mathcal{E}}$:

$\alpha^{i*} = (x_t^{i*}, \xi_{t+1}^{i*}, \zeta_{t+1}^{i*})$, for all $t \geq 1$ and for all $i \in \mathcal{I}_t$,

$\alpha^{i*} = (\xi_1^{i*}, \zeta_2^{i*})$, for all $i \in \mathcal{I}_0$.

□

# Appendix

## Boundedness of $\mathcal{A}(\tilde{\mathcal{E}})$

Let $e \in \prod_{t \in \mathbb{N}^*} \mathbb{R}_+^L$ be defined by $\tilde{e}_t = \sum_{i \in \mathcal{I}_t \cup \mathcal{I}_{t-1}} \tilde{e}_t^i$ and $e' \in \prod_{t \in \mathbb{N}^*} \mathbb{R}_+^L$ such that $e' \geq \tilde{e}$. Let $(\chi^i) \in \mathcal{A}(\tilde{\mathcal{E}})$, with $\chi^i = (x_t^i, \xi_{t+1}^i, \zeta_{t+2}^i)$, then for all $t = 1, 2 \ldots$,

$$\sum_{i \in \mathcal{I}_0} \xi_1^i + \sum_{i \in \mathcal{I}_1} x_1^i \leq e'_1$$

$$\sum_{i\in\mathcal{I}_{t-2}} \zeta_t^i + \sum_{i\in\mathcal{I}_{t-1}} \xi_t^i + \sum_{i\in\mathcal{I}_t} x_t^i \le e_t',\ t > 1$$

For the first period, define $M_1 := e_1'$. Since $\xi_1^i \ge 0$ and $x_1^i \ge 0$, we have $0 \le x_1^i \le M_1$ for $i \in \mathcal{I}_0$, and $0 \le \xi_1^i \le M_1$ for $i \in \mathcal{I}_1$.
For the second period, let $M_2 := e_2' + \Gamma^1(M_1)$.
For $i \in \mathcal{I}_0$, since $\zeta_2^i = -\Gamma^1(\xi_1^i)$, we have: $-\Gamma^1(M_1) \le \zeta_2^i \le 0$.
By definition, we know that:

$$\sum_{i\in\mathcal{I}_0} \zeta_2^i + \sum_{i\in\mathcal{I}_1} \xi_2^i \ge \sum_{i\in\mathcal{I}_0} -\Gamma^1(\xi_1^i) - \sum_{i\in\mathcal{I}_1} \Gamma^1(x_1^i) \ge -\Gamma^1(e_1')$$

Thus, for $i \in \mathcal{I}_2$, $0 \le x_2^i \le \sum_{i\in\mathcal{I}_2} x_2^i \le e_2' - \sum_{i\in\mathcal{I}_0} \zeta_2^i - \sum_{i\in\mathcal{I}_1} \xi_2^i \le e_2' + \Gamma^1(e_1')$, that is: $0 \le x_2^i \le M_2$.
For $i \in \mathcal{I}_1$, it is clear that $\zeta_2^i \ge -\Gamma^1(e_1')$. Futhermore,

$$\zeta_2^i + \sum_{i'\in\mathcal{I}_1, i'\neq i} \zeta_2^{i'} \le e_2' - \sum_{i\in\mathcal{I}_0} \zeta_2^i$$

But,

$$\sum_{i'\in\mathcal{I}_1, i'\neq i} \zeta_2^{i'} + \sum_{i\in\mathcal{I}_0} \zeta_2^i \ge -\Gamma\Big(\sum_{i'\in\mathcal{I}_1, i'\neq i} x_1^i\Big) - \Gamma\Big(\sum_{i\in\mathcal{I}_0} \xi_1^i\Big) \ge -\Gamma(e_1')$$

Thus, for $i \in \mathcal{I}_1$, $-\Gamma^1(M_1) \le \zeta_2^i \le e_2' + \Gamma(e_1') = M_2$.
For period $t \ge 3$, we recursively proceed with the same reasoning to prove that the sequence of nonnegative vectors $(M_t)_{t\ge 1}$, defined by $M_t = e_t' + \Gamma^{t-1}(M_{t-1} + \Gamma^{t-2}(M_{t-2}) \cdots + \Gamma^1(M_1))$, where $M_1 = e_1'$satisfies the desired inequalities.
□

## Proof of Lemma 1

There is no modification concerning the budget constraints feasibility, we just have to look at Condition (b) for the period $\tau - 1$ in the definition of a pseudo-equilibrium. Since $(\bar{p}^*, (\bar{a}^{i*}))$ is a pseudo-equilibrium in the economy $\mathcal{E}_{\bar{\tau}}$ and $\bar{\tau} - 1 > \tau - 1$, one has:

$$\sum_{i\in\mathcal{I}_0^{\bar{\tau}-2}} \bar{a}_{\tau-1}^{i*} = \sum_{i\in\mathcal{I}_0^{\bar{\tau}-2}} \tilde{e}_{\tau-1}^{\bar{\tau}i}$$

Considering the definition of $X^{\bar{\tau}i}$, for all $i \in \cup_{t=\tau}^{\bar{\tau}-1}\mathcal{I}_t$, $\bar{a}_{\tau-1}^{i*} = 0$. From the definition of $\tilde{e}^{\bar{\tau}i}$, for all $i \in \cup_{t=\tau}^{\bar{\tau}-1}\mathcal{I}_t$, $\tilde{e}_{\tau-1}^{\bar{\tau}i} = 0$. So, one deduces that:

$$\sum_{i\in\mathcal{I}_0^{\bar{\tau}-2}} \bar{a}_{\tau-1}^{i*} = \sum_{i\in\mathcal{I}_0^{\tau-2}} \bar{a}_{\tau-1}^{i*} + \sum_{i\in\mathcal{I}_{\tau-1}} \bar{a}_{\tau-1}^{i*} = \sum_{i\in\mathcal{I}_0^{\tau-2}} \tilde{e}_{\tau-1}^{\bar{\tau}i} + \sum_{i\in\mathcal{I}_{\tau-1}} \tilde{e}_{\tau-1}$$

and since $\sum_{i\in\mathcal{I}_{\tau-1}} \bar{a}^{i*}_{\tau-1} \geq 0$, we have:

$$\sum_{i\in\mathcal{I}_0^{\tau-2}} \bar{a}^{i*}_{\tau-1} \leq \sum_{i\in\mathcal{I}_0^{\tau-2}} \tilde{e}^{\bar{\tau}i}_{\tau-1} + \sum_{i\in\mathcal{I}_{\tau-1}} \tilde{e}_{\tau-1}$$

So we get Condition (b) for the period $\tau-1$ since $\bar{a}^{i*}_{\tau-1} = \hat{a}^{i*}_{\tau-1}$ and $\tilde{e}^{\bar{\tau}i}_{\tau-1} = \tilde{e}^{\tau i}_{\tau-1}$ for all $i \in \mathcal{I}_0^{\bar{\tau}-2}$ and $\tilde{e}^{\bar{\tau}i}_{\tau-1} = \tilde{e}^{i}_{\tau-1}$ for all $i \in \mathcal{I}_{\tau-1}$.
□

## Proof of Lemma 2

The first part comes from the strict monotonicity of the utility of agent $i \in \mathcal{I}_t$, for all $t = 1, 2 \ldots \tau - 1$, as mentionned in Assumption C.

We normalize a non zero equilibrium price $p^*$ so that $\sum_{t=1}^{\tau} \sum_{\ell\in\mathcal{L}} p^*_{t\ell} = 1$. Let us consider a sequence of pseudo-equilibria $(p^\nu, (a^{i\nu}))$ that converges to $(\bar{p}, (\bar{a}^i))$. We prove that $(\bar{p}, (\bar{a}^i))$ is also a pseudo-equilibrium.

We easily establish that $(\bar{p}, (\bar{a}^i))$ satisfies Condition (b) in Definition 3 of pseudo-equilibrium. So it remains to show that Condition (a) is also satisfied.

Denote by $(w^{i\nu})$ the associated wealth sequence and by $\bar{w}^i$ its limit. One easily shows that the budget constraint is satisfied by $\bar{a}^i$. If $\bar{p} \cdot a^i < \bar{w}^i$, then for $\nu$ large enough, $p^\nu \cdot a^i \leq w^{i\nu}$. But this implies that $\tilde{u}^i(a^i) \leq \tilde{u}^i(a^{i\nu})$, and by the continuity of $\tilde{u}^i$, $\tilde{u}^i(a^i) \leq \tilde{u}^i(\bar{a}^i)$. Thus $(\bar{p}, (\bar{a}^i))$ satisfies Condition (a) in Definition 5 of a quasi-equilibrium. Thus $(\bar{p}, (\bar{a}^i))$ is actually a "pseudo-quasi-equilibrium". But thanks to the irreducibility condition, we can discard the possibility of minimal wealth at any quasi-equilibrium price. Thus each agent is an utility maximizer at any quasi-equilibrium price. □

## Proof of Lemma 3

We have established that for all $T \geq 2$, there exists a pseudo-equilibrium $(p^T, (a^{iT}))$ of the truncated economy $\tilde{\mathcal{E}}_T$. Since $p^T_1 \neq 0$, we normalize $p^T$ so that $\sum_{\ell\in\mathcal{L}} p^T_{1\ell} = 1$.

We extend the price and the allocations to the whole space $\prod_{t=1}^{\infty} \mathbb{R}^L$ by adding zeros for the missing components without modifying the notations. So, now the sequences $(p^T)$, $(a^{iT})$ are in $\prod_{t=1}^{\infty} \mathbb{R}^L$.

If Lemma 3 is not true, then there exist $\bar{t}$ and an increasing sequence $(T^\nu)$ such that $p^{T^\nu}_{\bar{t}} \geq \nu \mathbf{1}$. Let $\tau > \bar{t} + 3$. We assume without any loss of generality that $T^\nu > \tau$ for all $\nu$.

Now we consider the restriction to the $\tau$ first period of the $T^\nu$-equilibrium $(p^{T^\nu}, (a^{iT^\nu}))$:

- for all $i \in \mathcal{I}_0^{\tau-2}$, $a^{i\nu}$ is the restriction of $a^{iT^\nu}$ to $\prod_{t=1}^{\tau} \mathbb{R}^L$;

- $p^\nu$ is the restriction of $p^{T^\nu}$ to $\prod_{t=1}^{\tau} \mathbb{R}^L$.

From Lemma 1 in the previous section, $(p^\nu, (x^{i\nu})))$ is a pseudo-equilibrium of the truncated economy $\mathcal{E}_\tau$. We now renormalize the price $p^\nu$ as follows:

$$\pi^\nu = \frac{1}{\sum_{t=1}^{\tau} \sum_{\ell \in \mathcal{L}} p_{t\ell}^\nu} p^\nu$$

Since $\pi^\nu$ is nonnegative, the sequence $\pi^\nu$ remains in the simplex of $\prod_{t=1}^{\tau} \mathbb{R}^L$, which is compact. From the boundedness of $\mathcal{A}(\tilde{\mathcal{E}}_\tau(e))$, the sequence $(a^{i\nu})$ remains in the compact subset $\mathcal{A}(\tilde{\mathcal{E}}_\tau(e))$. So the sequence $(\pi^\nu, (a^{i\nu})$ has a cluster point $(\bar{\pi}, (\bar{a}^i))$. From Lemma 2, $(\bar{\pi}, (\bar{a}^i))$ is also a pseudo-equilibrium of the truncated economy $\tilde{\mathcal{E}}_\tau$. But $\bar{\pi}_1 = 0$ since

$$\sum_{t=1}^{\tau} \sum_{\ell \in \mathcal{L}} p_{t\ell}^\nu \geq \sum_{\ell \in \mathcal{L}} p_{\bar{t}\ell}^\nu \geq \nu L$$

converges to $+\infty$ and $0 \leq p_{1\ell}^\nu \leq 1$ for all $\ell \in L$. Hence we get a contradiction since Lemma 2 shows that for all $t = 1, \ldots, \tau$, $\bar{\pi}_t \neq 0$.

□

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